Dilaton black holes in grand canonical ensemble near the extreme state

O.B.Zaslavskii

Department of Physics, Kharkov State University, Svobody Sq.4, Kharkov 310077, Ukraine

E-mail: oleg.b.zaslavskii@univer.kharkov.ua

Dilaton black holes with a pure electric charge are considered in the frame work of a grand canonical ensemble near the extreme state. It is shown that there exists such a subset of boundary data that the Hawking temperature smoothly goes to zero to an infinite value of a horizon radius but the horizon area and entropy are finite and differ from zero. In string theory the existence of a horizon in the extreme limit is due to the finiteness of a system only.

04.70Dy, 97.60Lf, 04.20 Jb

The study of the nature of the extreme state is one of the "hottest" areas in current researches on black hole physics. Especially this concerns the possibility of their thermodynamic description and behavior of entropy and temperature near such a state. In this respect "ordinary" black holes, for example Reissner-Nordström spacetimes (RN), and dilaton ones are in some sense complementary to each other. For a RN black hole with mass m and charge q the Hawking temperature $T_H \propto \sqrt{m^2-q^2}$goes to zero smoothly when the ratio $r_+/r \to 1$ ($r_+ = m+\sqrt{m^2-q^2}$ is the radius of the event horizon, $r_- = m-\sqrt{m^2-q^2}$ is that of an inner one). In so doing, the entropy $S = A/4$ (A is the horizon area) remains finite nonzero quantity πr_+^2. On the other hand, for dilaton black holes [1], [2], [3] $T_H = (4\pi r_+)^{-1}$, while $S = \pi r_+^2(1-r_{-/}r_+)$ where r_+ and r_- are parameters of the metric whose explicit form is listed below. Thus, for RN black holes there exists the limit $r_{+/}r_- \to 1$ in which $T_H \to 0$, $S \neq 0$ and for dilaton black holes there exists the limit in which $T_H \neq 0$, $S \to 0$ (the extreme state).

Meanwhile, it has been shown recently [4], [5] that careful treatment in the framework of

gravitational thermodynamics which takes into account properly the finiteness of a system size displays qualitatively new features in the geometry and thermodynamics of RN black holes near the extreme state. In particular, it turned out that the extreme state can be achieved at *finite temperature* T that determines properties of a canonical and grand canonical ensemble and differs from T_H by blueshifting factor [6]. It is natural to pose the same problem for dilaton black holes and examine the possible role for them of the finiteness of a system.

In the present paper I show that this role is highly nontrivial. On the basis of treatment in an infinite space it was believed that in the extreme limit the horizon inevitably becomes singular with zero surface area [3]. However, it turns out that there exist such configurations that the limit $r_+/r_- \rightarrow 1$ can be achieved with the a *regular* horizon and *finite entropy.* It is essential that this result is obtained in a model-independent way in view of primary principles of gravitational thermodynamics. It occurs even in the simplest case of a pure electrically charged holes to which we restrict ourselves in this issue. (Therefore, this result should not be confused with those for extreme dilaton black holes in supersymmetric theories [7] where the property $S \neq 0$ for them is due to the simultaneous existence of both electrical and magnetic charges.) As far as temperature is concerned, it turns out that $T_H \rightarrow 0$ smoothly but T remains finite.

Apart from this, I discuss briefly the example from Ref. [3] (where black holes with a pure magnetic charge are considered) for the metric in a string theory. Here gravitational thermodynamics predicts that in addition to a bottomless hole without a horizon described in [3] a black hole also may exist in the limit $r_+ = r_-$.

Both results rely strongly on the finite size of a system which enters as one of boundary data in their complete set in the grand canonical approach which from methodical viewpoint generalized the approach of Ref. [8] developed for black holes in the Einstein-Maxwell theory to the case of dilaton ones.

It follows from [1]- [3] that the Euclidean metric of dilaton black holes with a pure electric charge (or with a pure magnetic one) reads

$$ds^2 = (1 - r_+/r)d\tau^2 + (1 - r_+/r)^{-1}dr^2 + R^2 d\omega^2 \tag{1}$$

where $d\omega^2$ is the metric on a sphere of an unit radius. For this spacetime

$$R^2 = r^2(1 - r_-/r),\ e^{2\varphi} = e^{2\varphi_0}(1 - r_-/r),\ F = Q/r^2 dt \wedge dr,\ Q^2 = \frac{r_+ r_-}{2} e^{2\varphi_0} \tag{2}$$

Here F is electromagnetic field, φ is a dilaton, the constant φ_0 for an infinite space coincides with $\varphi(r = \infty)$. To avoid possible confusion with the sign of φ,one should bear in mind that we discuss the case of a black hole with a pure electric charge whereas in Ref. [3] a charge is pure magnetic. Henceforth, a black hole is supposed to be enclosed in a cavity. Then the state of a system in a grand canonical ensemble is determined by boundary data [8] whose complete set contains in our case β, ϕ_B, φ_B, R_B. Here β is the inverse Tolman temperature on a boundary, ϕ_B is the difference of potentials between a horizon and boundary, φ_B is a dilaton value on a boundary. The key moment consists in that just R_B determines the metric induced on a boundary surface and its area $4\pi R_B^2$ and for this reason R_B but not r_B must enter the set of boundary data for a thermal ensemble. (A radial coordinate r is in fact an auxiliary variable in terms of which the surface area should be expressed. One could, in principle, rescale r and obtain for it another boundary value with the same R_B.)

Following primary principles of gravitational thermodynamics [6] which takes into account properly space inhomogeneity of gravitating systems one should write down relevant quantities (the inverse temperature, surface area and the value of dilaton field on our case) as functions of coordinates and equate them with their boundary values. As a result, we obtain equations

$$\beta = 4\pi r_+(1 - r_+/r_B)^{1/2} \tag{3}$$

$$R_B = r_B(1 - r_-/r_B)^{1/2} \tag{4}$$

$$\phi = Q(r_+^{-1} - r_B^{-1})(1 - r_+/r_B)^{-1/2} \tag{5}$$

Here eq. 3 takes into account that β is blueshifted according to the Tolman formula with the Hawking temperature $T_H = (4\pi r_+)^{-1}$. Eq. 5 plays the similar role for the potential (cf.

with Einstein-Maxwell black holes [8]). A charge Q in terms of boundary data φ_B instead of φ_0 equals

$$Q^2 = \frac{r_+ r_-}{2} e^{2\varphi_B} (1 - r_-/r_B)^{-1} \tag{6}$$

After substitution of eq.6 into eq.5 we have three equations for three variables r_B, r_+ and r_- in terms of boundary data $\beta, R_B, \varphi_B, \phi_B$. As is shown below, ϕ_B and φ_B enter the set of boundary data in a single combination, so we have three equations for three variables. (In the simplest case of Schwarzschild black hole $R = r$ and one would have only one eq.3 from which the horizon radius is to be defined as $r_+ = r_+(r_B, \beta)$ [6].) Once r_B, r_+ and r_- are determined in terms of boundary data one can find, for example, the horizon area in terms of these data.

Now we will show that among all possible boundary data there exists such a subset with *finite* β, ϕ_B and R_B that the ratio $r_+/r_- \to 1$ in such a way that the following relations hold

$$r_+/r_B \to 1,\ r_+, r_-, r_B \to \infty,\ T_H \to 0,\ A \neq 0 \tag{7}$$

(I recall that r_B in contrast to R_B is not a fixed parameter but a variable to be determined from eqs. 3-6.)

It is convenient to introduce dimensionless variables $x = r_+/r_B$, $y = r_-/r_B$, $\sigma = \beta/4\pi R_B$. Then eqs. 3-6 take the form

$$\sigma = x(1-x)^{1/2} z^{-1},\ z = (1-y)^{1/2},\ \alpha \equiv \phi_B e^{-\varphi_B} \sqrt{2} = [y(1-x)/x(1-y)]^{1/2} \tag{8}$$

Note that either Q or φ_0 do not enter 8 explicitly as they are eliminated according to 5, 6.

It follows from 8 that

$$\alpha^2 = \sigma^2 x^{-3} + 1 - x^{-1} \tag{9}$$

Let $\alpha = \sigma$. Then eq. 9 has the root $x = 1$. In so doing, $y = 1$, $z = 0$ according to 8. If $x = 1 - \varepsilon$ and $y = 1 - \delta$ where $\varepsilon, \delta << 1$ eq. (8) shows the law according to which the

point (x,y) approaches the limit under discussion: $\varepsilon/\delta = \alpha^2 = \sigma^2$. Returning to dimension variables we see that conditions 7 are indeed satisfied.

Solutions found above can be also obtained with the help of the Euclidean action. The crucial role in this approach is played by the finiteness of a system as was first demonstrated in [6] for Schwarzschild black holes. In particular, there exists such a range of boundary data that Euclidean action not only has a local minimum as a function of a horizon radius but also this minimum is global, so $I < 0$ (it is assumed that $I = 0$ for a hot flat space). It means that for a corresponding set of boundary data a black hole spacetime is a favorable phase, so placing a system into a box with spherical walls can stabilize such a state. This is intimately connected with the space inhomogeneity of self-gravitating systems which reveals itself in the crucial role of boundary conditions (first of all, it makes a canonical or grand canonical ensemble for black holes well defined and, in particular, leads to the possibility of a positive heat capacity for a Schwarzschild black hole which was believed earlier to be only unstable). Below we will see that allowance for the finiteness of a system for dilaton holes leads to the possibility of a stable phase for the problem under discussion as well.

The Euclidean action reads

$$I = \beta E - S - \beta \phi_B q \tag{10}$$

Here the constant q appears in the analog of the Gauss law which can be obtained by integrating the field equation $[F_{01}R^2 \exp(-2\varphi_0)], r = 0$. This constant $q = Q\exp(-2\varphi_0)$ differs from Q due to coupling between the electromagnetic and electric fields. The entropy $S = A/4$, E is the energy. The energy density $\varepsilon = (k - k_0)/8\pi$ [9], [10] where k is the extrinsic curvature of the boundary in three-dimensional spatial metric of slices $\tau = const$. The constant k_0 is chosen to make $I = 0$ for a hot flat space. Calculating k and using dimensionless variables $x = r_+/r$ and $y = r_-/r$ one obtains the expression

$$J \equiv I/4\pi R_B^2 = \sigma[1 - (1 - \frac{y}{2})(1-x)^{1/2}(1-y)^{-1/2}] - \frac{(x^2 - xy)}{4(1-y)} - \frac{\sigma\alpha}{2}(xy)^{1/2} \tag{11}$$

It is convenient to introduce new variables $\eta = (1-x)^{1/2}(1-y)^{-1/2}$ and $\delta = 1 - y$. Then for $\alpha = \sigma$ we get dropping all terms of the third order and higher in δ :

$$J=2(\eta-\sigma)^2+\delta[(\eta-\sigma)^2+\eta^2(\sigma^2-\eta^2)]+\frac{\sigma^2\delta^2(1-\eta^2)^2}{4} \qquad (12)$$

It follows from 12 that $\eta=\sigma$, $\delta=0$ $(y=1)$ is indeed a local extremum. If $\sigma^2<\sqrt{2}-1$ this point is a minimum. If, apart from this, $\sigma^2<\frac{3}{2}-\sqrt{2}$ then $J<0$, so this minimum is global, i.e. a black hole is thermodynamically favorable phase as compared to a hot flat space. Thus, there exists a finite range of σ within which the found solution is stable either locally or globally.

Now we will find the form of the metric in the state under discussion. As $r_+<r<r_B$ and $r_+/r_B\rightarrow 1$ the coordinate r becomes singular not only near the horizon but for the *whole* Euclidean manifold. It is convenient to use instead of r the proper distance l from the horizon l. In the limit under consideration $r\rightarrow r_B$ and $r-r_+=l^2/4r_B$ where $r_B\rightarrow\infty$. Let the Euclidean time be normalized according to $0\leq\tau\leq 2\pi$. Then expressing r in terms of l we get after simple calculations with eqs. 3, 8 taken into account:

$$ds^2=d\tau^2l^2+dl^2+[R_B^2+\frac{(l^2-l_B^2)}{4}]d\omega^2 \qquad (13)$$

It follows from the Tolman formula that $\beta=2\pi l_B$ $(l_B=2\sigma R_B)$ and we see that the horizon area $A=4\pi R_B^2(1-\sigma^2)$ is a finite nonzero quantity. (Note that just R_B but not r_B enters the final expression for A in accordance with what was said above about relation between these quantities.)

The metric 13 can be also rewritten in the Schwarzschild-like form

$$ds^2=4(\rho^2-\rho_0^2)d\tau^2+4d\rho^2(1-\rho_0^2/\rho^2)^{-1}+\rho^2d\omega^2 \qquad (14)$$

where $\rho^2=R_B^2+(l^2-l_B^2)/4$, $\rho_0^2=R_B^2-l_H^2/4$.

Let us discuss now another case - metric in string theory that was considered in Ref. [3] with a pure magnetic charge. The relevant metric $\tilde{g}_{\mu\nu}$ is obtained from the original one $g_{\mu\nu}$ by conformal transformation with the factor $e^{2\varphi}$. According to 1, 2 this factor in terms of boundary data equals

$$\exp(2\varphi)=\exp(2\varphi_B)(1-r_-/r_B)^{-1}(1-r_-/r) \qquad (15)$$

Then omitting constant terms we obtain

$$d\tilde{s}^2 = (1 - r_+/r)(1 - r_-/r)^{-1}d\tau^2 + [(1 - r_+/r)(1 - r_-/r)]^{-1}dr^2 + r^2 d\omega^2 \tag{16}$$

where r takes finite values in the range $r_+ \leq r \leq r_B$. It was pointed out in [3] that in the extreme limit $r_+ = r_-$ the metric describes a bottomless hole ($l = \infty$ for any $r > r_+$) without an event horizon.

However, it is worth paying attention that this is indeed the case only if one put $r_+ = r_-$ right from the beginning. If this limit is achieved from a topological sector of black holes in the manner described above the situation is qualitatively different. Now according to eq.2 and eq.7 the factor $\exp(2\varphi)$ is equal to $\exp(2\varphi_B)R^2/R_B^2$. It is finite and nonzero everywhere (whereas it turned to the zero at the horizon $r = r_+ = r_-$ in the previous case), so the properties of the metric $\tilde{g}_{\mu\nu}$ in string theory are qualitatively similar to whose of the original one $g_{\mu\nu}$. In particular, the horizon does not disappear.

It is instructive to compare relationship between extreme and non-extreme cases for RN and dilaton black holes in terms of geometrical characteristics. In the first case the geometrical property which signals about extremality consists in $l = \infty$ for any point with $r > r_+$ outside the horizon. Therefore, it turned out rather unexpectedly that in the framework of the grand canonical ensemble the limiting transition can be performed in such a way that l remains finite when $r_+/r_- \to 1$ [4], [5]. On the other hand, it *would seem* obvious from results of [3] that for dilaton black holes with a pure electric (or pure magnetic) charge $A \to 0$ in the extreme limit $r_+/r_- = 1$. None the less, as it follows from the results of the present paper, for the particular class of boundary data this quantity tends to the finite nonzero limit.

Thus, in both cases careful treatment in terms of thermodynamics shows that the appropriate choice of boundary data affects geometry and even topology crucially. It is worth stressing that the key point here is the finiteness of a system. Otherwise the solutions considered above would have been lost. The situation is especially pronounced in the example from string theory when the possibility of the existence of a horizon is due to the finiteness

of a system entirely.

The results obtained in this paper can be extended to the case of more general coupling between dilaton and electromagnetic field in a straightforward manner. What is more the mechanism by means of which thermodynamic and geometrical properties are overlapped near the extreme state for spacetimes similar to 1 is insensitive to subtle details of a field theory within which a general form 1 is obtained.

---

[1] G.W.Gibbons, Nucl.Phys.**B 207**, 337 (1982)

[2] G.W.Gibbons and K.Maeda, Nucl.Phys.**B 298**, 741 (1988)

[3] D.Garfinkle, G.T.Horowitz, and A.Strominger, Phys.Rev. **D43**, 3140 (1991); Erratum **D 45**, 3888 (1992)

[4] O.B.Zaslavskii, Phys.Rev.Lett. **76**, 2211 (1996)

[5] O.B.Zaslavskii, Phys.Rev. D**56**, 2188 (1997)

[6] J.W.York, Jr., Phys.Rev. D**33,** 2092 (1986).

[7] R.Kallosh, A.Linde, T.Ortin, A.Peet, and A.V.Proeyen, Phys.Rev. **D 46**, 5278 (1992)

[8] H.W.Braden, J.D.Brown, B.F.Whiting, and J.W.York, Jr., Phys.Rev. **D 42**, 3376 (1990)

[9] O.B.Zaslavskii, Phys.Lett. **A152**, 463 (1991)

[10] J.D.Brown and J.W.York, Jr., Phys.Rev. **D 47**, 1420 (1993)